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POWERFUL IDEALS, STRONGLY PRIMARY IDEALS, ALMOST PSEUDO-VALUATION DOMAINS, AND CONDUCIVE DOMAINS

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ABSTRACT

Let R be a domain with quotient field K, and let I be an ideal of R. We say that I is powerful (strongly primary) if whenever $x, y \in K$ and $xy \in I$, we have $x \in R$ or $y \in R$ ($x \in I$ or $y^n \in I$ for some $n \geq 1$). We show that an ideal with either of these properties is comparable to every prime ideal of R, that an ideal is strongly primary $\Leftrightarrow$ it is a primary ideal in some valuation overring of R, and that R admits a powerful ideal $\Leftrightarrow$ R admits a strongly primary ideal $\Leftrightarrow$ R is conducive in the sense of Dobbs-Fedder. Finally, we study domains each of whose prime ideals is strongly primary.

INTRODUCTION

Throughout this work R will denote an integral domain with quotient field K. Recall from [10] that a prime ideal P of R is said to be *strongly prime* if, whenever $xy \in P$ for elements $x, y \in K$, we have $x \in P$ or $y \in P$. In this paper, we consider two generalizations of this concept. We define a nonzero ideal I of R to be *powerful* if, whenever $xy \in I$ for elements $x, y \in K$, we have $x \in R$ or $y \in R$. It is easy to see that R itself is powerful $\Leftrightarrow$ R is a valuation domain. In the first section, we show that a powerful prime ideal is strongly prime. We also show that if I is a proper powerful ideal of R, then its radical $\mathrm{Rad}(I)$ is prime in general and strongly prime when R is seminormal [Propositions 1.9 and 1.12]. Moreover, a powerful ideal I is comparable to every nonzero prime of R, from which it follows that the prime ideals contained in $\mathrm{Rad}(I)$ are linearly ordered [Theorem 1.5]. The remainder of the first section is devoted to a study of overrings. We show in Proposition 1.17 that if R admits a powerful ideal and T is an overring of R, then R and T share an ideal which is powerful in both rings. Conversely, in Proposition 1.18, we prove that if T is an overring of R such that R and T share a common ideal J which is powerful in T, then J^3 is a powerful ideal of R.

As another generalization of the notion of "strongly prime," in Sec. 2 we define an ideal I of R to be *strongly primary* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$. Simple examples show that "powerful" and "strongly primary" are different notions. A proper strongly primary ideal of R is clearly primary, and we observe that the converse is true in a valuation domain [Proposition 2.1]. In fact, noticing that the property of being strongly primary is independent of the domain in which I happens to be an ideal, this characterizes whether a given ideal is strongly primary. More precisely, we show in Theorem 2.11 that an ideal of I of R is strongly primary $\Leftrightarrow$ I is a primary ideal in some valuation overring of R. We also show that a strongly primary ideal of R is comparable to every radical ideal of R [Theorem 2.8] and that a proper strongly primary ideal I of a seminormal domain is powerful with $\mathrm{Rad}(I)$ strongly prime [Theorem 2.4].

Section 3 is devoted to a study of a generalization of pseudo-valuation domains (PVDs). Recall from [10] that a PVD is a domain in which each nonzero ideal is strongly prime and that PVDs are characterized as quasilocal domains $(R : M)$ with the property that $(M : M)$ is a valuation domain with maximal ideal M. We define an *almost pseudo-valuation domain* (APVD) to be a domain each of whose prime ideals is strongly primary. The main result of this section then characterizes APVDs as quasilocal domains (R, M) such that $(M : M)$ is a valuation domain with M primary to the maximal ideal of $(M : M)$. We also consider overrings of an APVD, and we prove that the integral closure of an APVD is a PVD.

Finally, in Sec. 4, we relate our work to the conducive domains of Dobbs-Fedder. In [7], they define a conducive domain to be a domain R such that for each overring $T \neq K$ of R, the conductor $(R : T)$ is nonzero. We show that a domain R is conducive $\Leftrightarrow$ R admits a powerful ideal $\Leftrightarrow$ R admits a strongly primary ideal. We also use our techniques to recover the Dobbs-Fedder characterization of conducive Prüfer domains and the Barucci-Dobbs-Fontana characterization of conducive Noetherian domains.

1. POWERFUL IDEALS

We begin with a simple but useful restatement of the definition of *powerful.*

Lemma 1.1. *An ideal I of R is powerful $\Leftrightarrow$ $x^{-1}I \subseteq R$ for each $x \in K\backslash R$.*

Proof. Assume that I is powerful, and let $x \in K\backslash R$. Then for $a \in I$ we have $xx^{-1}a = a \in I$, whence $x^{-1}a \in R$. For the converse, let $yz \in I$, $y, z \in K$. Suppose $y \notin R$. Then $z = y^{-1}yz \in y^{-1}I \subseteq R$, as desired. □

As an easy consequence of the lemma, we obtain the fact that powerfulness is preserved upon passage to homomorphic images.

Proposition 1.2. *Let I be a powerful ideal of R, and let Q be a prime ideal of R which is properly contained in I. Then I/Q is powerful in R/Q.*

Proof. Let $\phi : R \to R/Q$ denote the canonical homomorphism. Suppose that $x = \phi(y)/\phi(z)$ is an element of the quotient field of R/Q with $x \notin R/Q$. Then $y/z \notin R$. Hence if $a \in I$, we have $(z/y)a \in R$, and it follows that $(\phi(z)/\phi(y))\phi(a) \in R/Q$. Thus $x^{-1}(I/Q) \subseteq R/Q$, as desired. □

Proposition 1.3. *A prime ideal of R is strongly prime $\Leftrightarrow$ it is powerful.*

Proof. Suppose that P is a powerful prime ideal of R. Let $xy \in P$ for some $x, y \in K$. Then $x^2y^2 \in P$. We may assume $x \notin R$ and $y \in R$. If $x^2 \in R$, then, since $x \notin R$, $x^2 \notin P$, and the fact that $x^2y^2 \in P$ then implies that $y^2 \in P$, whence $y \in P$. If $x^2 \notin R$, then, since $(y^2/xy)x^2 \in P$, we have $y^2/xy \in R$. Hence $y^2 = (y^2/xy)xy \in P$, and again we have $y \in P$. The converse is trivial. □

It is known that if $Q \subseteq P$ are prime ideals of R such that P is strongly prime, then Q is strongly prime [1, Proposition 4.8]. By Proposition 1.3, the following result, whose proof is trivial, generalizes this fact.

Proposition 1.4. *If $J \subseteq I$ are ideals of R with I powerful, then J is also powerful.*

Theorem 1.5. *Let I be a powerful ideal of R.*

(1) If J is an ideal of R, then either $J \subseteq I$ or $I^2 \subseteq J$.
(2) If J is a prime ideal of R, then I and J are comparable.
(3) The prime ideals of R contained in $\text{Rad}(I)$ *are linearly ordered.*

Proof. To prove (1), suppose that J is an ideal of R with $J \not\subseteq I$. Choose $a \in J \backslash I$, and let $b, c \in I$. Then $(bc/a)(a/b) \in I$, and since I is powerful with $a/b \notin R$, we have $bc/a \in R$. Hence $bc \in aR \subseteq J$, as desired. Statement (2) is immediate from (1). For (3), let P, Q be prime ideals properly contained in $\text{Rad}(I)$. Then P and Q are contained in I and are therefore powerful by Proposition 1.4. Hence they are comparable by (2). □

The following result is an easy consequence of Proposition 1.3 and Theorem 1.5.

Corollary 1.6. *A domain R is a PVD $\Leftrightarrow$ some maximal ideal of R is powerful.*

Proposition 1.7. *If R contains a powerful ideal, then R contains a unique largest powerful ideal.*

Proof. It suffices to show that the sum of powerful ideals is again powerful. Thus let $\{I_\alpha\}$ denote a family of powerful ideals of R. If $x \in K \backslash R$, then $x^{-1} I_\alpha \subseteq R$ for each α by Lemma 1.1. Hence $x^{-1} \sum_\alpha I_\alpha \subseteq R$, and we have that $\sum_\alpha I_\alpha$ is powerful, again by Lemma 1.1. □

Our next result generalizes [10, Proposition 2.4].

Proposition 1.8. *If I is a proper powerful ideal of R, then $P = \cap_{k=0}^{\infty} I^k$ is a strongly prime ideal.*

Proof. It suffices by Propositions 1.3 and 1.4 to show that P is prime. Let $xy \in P$ with $x \in R \backslash P$. Then $x \notin I^n$ for some $n > 0$, whence by Theorem 1.5, $I^{2n} \subseteq xR$. Hence for each $k > 0$, we have $xy \in P \subseteq I^{2n+k} \subseteq xI^k$. Thus $y \in I^k$ for each $k > 0$. It follows that $y \in P$. □

Proposition 1.9. *Let I be a powerful ideal of R. If $x, y \in K$ and $xy \in$ $\text{Rad}(I)$, then there is a positive integer m such that either $x^m \in I$ or $y^m \in I$. In particular, if I is a proper powerful ideal, then* $\text{Rad}(I)$ *is prime.*

Proof. We have $(xy)^n \in I$ for some $n > 0$. Hence $(x^{3n}/x^ny^n)\,(y^{3n}/x^ny^n) = x^ny^n \in I$. Since I is powerful, either $x^{3n}/x^ny^n \in R$ or $y^{3n}/x^ny^n \in R$, whence either $x^{3n} \in I$ or $y^{3n} \in I$. □

In spite of Proposition 1.9, the radical of a powerful ideal need not be powerful, as the following example shows.

Example 1.10. Let $V = k + M$ be a rank one discrete valuation domain, where k is a field and $M = tV$ is the maximal ideal of V, and let $R = k + M^2$. Claim: M^3 is a powerful ideal of R. To see this, let $xy \in M^3$, with $x, y \in K$ (the common quotient field of R and V). We may write $x = ut^n, y = vt^m$, where u, v are units of V and n, m are integers. Since $xy \in M^3$, we must have $n + m \geq 3$. Hence either $n \geq 2$ or $m \geq 2$, say $n \geq 2$. Then $x = ut^n \in M^2 \subseteq R$. This proves the claim. However, (in R) $\mathrm{Rad}(M^3) = M^2$ is not powerful since $t^2 \in M^2$ but $t \notin R$.

We prove below that in a seminormal domain the radical of a powerful ideal is powerful. First, we need a lemma.

Lemma 1.11. *Let I be a powerful ideal of R. If $x \in K$ and $x^n \in I$ for some $n > 0$, then $x^{n+k} \in R$ for each $k \geq 0$.*

Proof. Let $e = \min\{m \geq 1 \mid x^m \in R\}$. Let k be a positive integer, and write $k = qe + r$ with $0 \leq r < e$. If $r = 0$, then it is easy to see that $x^{n+k} \in R$. Suppose that $r > 0$. We have $x^{e-r}x^{qe+n+r} = x^nx^{(q+1)e} \in I$. Since $x^{e-r} \notin R$, we have $x^{n+k} = x^{qe+n+r} \in R$, as desired. □

Now recall from [2] that a radical ideal J of R is said to be *strongly radical* if $x \in K$ and $x^n \in J$ for some $n > 0$ implies that $x \in J$. We also recall that R is seminormal if $x \in R$ whenever $x^n \in R$ for all sufficiently large n.

Proposition 1.12. *Let I be a proper powerful ideal of R. Then* $\mathrm{Rad}(I)$ *is powerful (and therefore strongly prime)* $\Leftrightarrow$ $\mathrm{Rad}(I)$ *is strongly radical. In particular, if R is seminormal, then,* $\mathrm{Rad}(I)$ *is strongly prime.*

Proof. It is easy to see that a powerful radical ideal must be strongly radical. Suppose that $\mathrm{Rad}(I)$ is strongly radical, and let $xy \in \mathrm{Rad}(I)$ with $x, y \in K$. Then by Proposition 1.9, we have $x^m \in I$ or $y^m \in I$ for some $m > 0$. We may suppose that $x^m \in I$. Then $x^m \in \mathrm{Rad}(I)$, whence $x \in R$, as desired. The "in particular" statement now follows from Lemma 1.11. □

Proposition 1.13. *Let I be a powerful ideal of R, and let T be an overring of R. Then IT is a powerful ideal of T. In particular, if $IT = T$, then T is a valuation domain.*

Proof. Let $x \in K \backslash T$. Then $x \notin R$, whence $x^{-1}I \subseteq R$ by Lemma 1.1. It follows that $x^{-1}IT \subseteq T$. Hence IT is powerful in T, again by Lemma 1.1. It is easy to see that a domain T is a valuation domain $\Leftrightarrow$ T is powerful as an ideal of T. □

Proposition 1.14. *Let I be a powerful ideal of R, and suppose that $P \subseteq I$ is a nonzero finitely generated prime ideal of R. Then R is a PVD with maximal ideal P.*

Proof. If P is not maximal, then R contains a nonunit x with $x \notin P$. Since P is strongly prime and $xx^{-1}P \subseteq P$ with $x \notin P$, we have $x^{-1}P \subseteq P$. Hence x^{-1} is integral over R, which is impossible. Thus P is maximal, and it follows that R is a PVD. □

We use R' to denote the integral closure of a domain R.

Theorem 1.15. *Suppose that R admits a powerful ideal I and that $M =$ Rad(I) is a maximal ideal of R. Then:*

(1) R is quasilocal with maximal ideal M.
(2) $IR' \subseteq M$, and therefore IR' is an ideal of R.
(3) R' is a PVD with maximal ideal $N =$ Rad(IR'), and hence $(N : N) = \{x \in K \mid xN \subseteq N\}$ is a valuation overring of R with maximal ideal N.

Proof. Statement (1) follows from Theorem 1.5. Now let $x \in R' \backslash R$. Note that since R' is integral over R, we must have $x^{-1} \notin R$. Hence $xI \subseteq R$ by Lemma 1.1. In fact, $xI \subseteq M$ (otherwise, $xI = R$, and $x^{-1} \in R$). It follows that $IR' \subseteq M$, proving (2). For (3), note that IR' is powerful in R' by Proposition 1.13. Hence $N = \text{Rad}(IR')$ is strongly prime in R' by Proposition 1.12. By (1), R' is quasilocal with maximal ideal N. Hence R' is a PVD by [10, Theorem 1.4], and $(N : N)$ is a valuation overring with maximal ideal N by [10, Theorem 2.10]. □

Corollary 1.16. *Let I be powerful in R, and let $P = \text{Rad}(I)$. Then $(R_P)'$ is a PVD with maximal ideal $N = \text{Rad}(I(R_P)')$. It follows that $(N : N)$ is a valuation overring of R with maximal ideal N.*

Proof. Note that P is prime by Proposition 1.19. By Proposition 1.13, IR_P is a powerful ideal of R_P, and the result now follows from Theorem 1.15. □

Proposition 1.17. *Let I be a powerful ideal of R, and let $T \neq K$ be an overring of R. Then R and T share an ideal which is powerful in both R and T. In fact:*

(1) *If $IT = T$, then $P = N \cap R$, where N is the maximal ideal of T, is a common ideal which is powerful in both rings.*
(2) *If $IT \neq T$, then I^2T is a common ideal, and I^3T is powerful in both rings.*

Proof. For (1), recall that T is a valuation domain by Proposition 1.13. By Theorem 1.5, I is comparable to P. The fact that $IT = T$ then implies that $P \subsetneq I$, whence P is powerful, and therefore strongly prime, in R. Note that PT is powerful in T by Proposition 1.13. We claim that $PT = P$. To verify this, let $x \in T \backslash R$. Clearly, $x^{-1} \notin P$. Hence, since $x^{-1}xP \subseteq P$ and P is strongly prime, we have $xP \subseteq P$, as claimed. For (2), let $x \in T \backslash R$. If $x^{-1} \notin R$, then $xI^2 \subseteq xI \subseteq R$ by Lemma 1.1. If $x^{-1} \in R$, then, by hypothesis, $x^{-1} \notin I$, whence $I^2 \subseteq x^{-1}R$ by Theorem 1.5. Hence, again, $xI^2 \subseteq R$. Thus I^2T is an ideal of R. Since $I^3T \subseteq I$, I^3T is powerful in R by Proposition 1.4, and I^3T is powerful in T by Proposition 1.13. □

Proposition 1.18. *Suppose that T is an overring of R and that R and T share the nonzero ideal J. If J is powerful in T, then J^3 is a powerful ideal of R.*

Proof. Let $x \in K \backslash R$. If $x \notin T$, then $x^{-1}J \subseteq T$ by Lemma 1.1. In this case, we have $x^{-1}J^3 \subseteq J^2T \subseteq R$. Now assume $x \in T$. Since $x \notin J$, we have $J^2 \subseteq xT$ by Theorem 1.5. Hence $x^{-1}J^3 \subseteq JT = J \subseteq R$, and the proof is complete. □

It is not difficult to show that, with the notation of Proposition 1.18, if T is a valuation domain, then J^2 is powerful in R. However, for general T, the third power is best possible, as the following example shows.

Example 1.19. Let k denote the field $\mathbb{Q}(\sqrt{2})$, and let $V = k[[X]] = k + M$, $M = Xk[[X]]$. Then let $T = \mathbb{Q} + M$, $J = XT$, and $R = \mathbb{Q} + J$. Then R and T share the ideal J, and, since T is a PVD, J is powerful in T. However, J^2 is not powerful in R, since $\sqrt{2}X \cdot \sqrt{2}X = 2X^2 \in J^2$, but $\sqrt{2}X \notin R$.

2. STRONGLY PRIMARY IDEALS

Proposition 2.1. *A primary ideal of a valuation domain is strongly primary.*

Proof. Let V be a valuation domain with quotient field K, let I be a primary ideal of V, let $x, y \in K$ with $xy \in I$, and suppose that $x \notin I$. If $x \notin V$, then $x^{-1} \in V$, and we have $y = x^{-1}xy \in I$. Hence we may as well assume that $x \in V$. Since $x = y^{-1}xy \notin I$, it follows that $y \in V$. Now, since $x, y \in V$ with I primary, we have $y^n \in I$ for some $n \geq 1$, as desired. □

Observe that if V is a valuation domain which is not rank one, then there are ideals which are not primary [9, Exercise 2, p. 292]. Since every ideal of a valuation domain is powerful, this shows that powerful ideals need not be (strongly) primary. Conversely, strongly primary ideals need not be powerful: In Example 1.10, M^2 is strongly primary but not powerful in R.

Notation 2.2. For a subset S of R, we define $E(S)$ by

$$E(S) = \{x \in K \mid x^n \notin S \text{ for each } n \geq 1\}.$$

The following lemma provides a useful restatement of the definition of strongly primary.

Lemma 2.3. *A nonzero ideal I of R is strongly primary if and only if $x^{-1}I \subseteq I$ for each $x \in E(I)$.*

Proof. If I is strongly primary and $x \in E(I)$, then the equation $xx^{-1}I = I$ implies that $x^{-1}I \subseteq I$. Conversely, if $yz \in I$ with $y, z \in K$ and $z \in E(I)$, then the hypothesis yields $y = z^{-1}yz \in z^{-1}I \subseteq I$, as desired. □

Theorem 2.4. *Let R be a seminormal domain. If I is a proper strongly primary ideal of R, then I is powerful, and* Rad(I) *is strongly prime. In particular, a prime ideal of R is strongly prime if and only if it is strongly primary.*

Proof. Let $x \in K\backslash R$; we shall show that $x^{-1}I \subseteq I$ (whence $x^{-1}I \subseteq R$). By Lemma 2.3, it suffices to show that $x^n \notin I$ for all $n \geq 1$. Suppose, on the contrary, that $x^r \in I$, with r minimal. It is then easy to see that $x^{-k} \notin I$ for each $k \geq 0$, that is, that $x^{-1} \in E(I)$. By Lemma 2.3, this implies that $x^{r+1} = xx^r \in xI \subseteq I$. By induction, we get $x^t \in I \subseteq R$ for each $t \geq r$. However, the seminormality of R then implies that $x \in R$, a contradiction. □

Proposition 2.5. *Let I be a proper strongly primary ideal of R, and let T be an overring of R. Then either $IT = T$ or $IT = I$.*

Proof. Assume $IT \neq T$, and pick $x \in T\backslash R$. If $x^{-n} \in I$ for some $n \geq 1$, then, since $IT \neq T$, x^{-n} is a nonunit of T, a contradiction. Hence $x^{-1} \in E(I)$, and we have $xI \subseteq I$ by Lemma 2.3. Thus $IT = I$. □

Recall that R' denotes the integral closure of the domain R.

Corollary 2.6. *If I is a proper strongly primary ideal of R, then $IR' = I$. Moreover, I^3 is powerful in both R and R'.*

Proof. The first conclusion follows from Proposition 2.5 and the lying over property of integral extensions. Since I is automatically strongly primary in

R', I is powerful in R' by Theorem 2.4. It follows that I^3 is powerful in R and R' by Proposition 1.18. □

Corollary 2.7. *If I is a proper strongly primary ideal of R, then $\cap_{n=1}^{\infty} I^n$ is a strongly prime ideal of R.*

Proof. This follows from Proposition 1.8 and the fact that I^3 is powerful. □

Theorem 2.8. *If I is a strongly primary ideal of R, then I is comparable to every radical ideal of R. Moreover, the prime ideals of R which are properly contained in I are strongly prime and linearly ordered.*

Proof. Let J be a radical ideal of R, and suppose that $I \not\subseteq J$. Choose $a \in I \backslash J$, and let $b \in J$. Since $(a^2/b)(b/a) = a \in I$ and $a^2/b \in E(R) \subseteq E(I)$, we have $b/a \in I$. Hence $J \subseteq I$, as desired. If P is a prime ideal which is properly contained in I, then, since I^3 is powerful by Corollary 2.6, and $P \subseteq I^3$, P is also powerful. Then P is strongly prime by Proposition 1.3. The rest follows from Theorem 1.5. □

The following result is an immediate consequence of Theorem 2.8.

Corollary 2.9. *If P is a prime ideal of R which is strongly primary but not strongly prime, then P is the only prime with this property.*

Theorem 2.10. *Let I be a strongly primary ideal of R, and let $T \neq K$ be an overring of R. Then R and T share a strongly primary ideal. In fact:*

(1) if $IT \neq T$, then $IT = I$ is a common strongly primary ideal;
(2) if $IT = T$, then T is strongly primary, and for each maximal ideal N of T, $N \cap R$ is a common strongly prime ideal of R and T.

Proof. Statement (1) follows from Proposition 2.5. Now assume $IT = T$. If $x \in E(T)$, then $x \in E(I)$, whence $x^{-1}I \subseteq I$. It follows that $x^{-1}T \subseteq T$, and T is strongly primary. Now let N be maximal in T, and let $P = N \cap R$. Then I is comparable to P by Theorem 2.8, and since $IT = T$, we must have $P \subsetneqq I$. By Theorem 2.8, P is strongly prime. It then follows from (1) that P is a common strongly prime ideal of R and T. □

Theorem 2.11. *Let I be a proper ideal of a domain R. Then the following statements are equivalent.*

(1) I is a strongly primary ideal of R.
(2) I is a primary ideal in some valuation overring of R.

(3) $V = (I : I)$ is a valuation domain, and I is (an ideal of V which is) primary to the maximal ideal of V.

Proof. (1) $\Rightarrow$ (3). Let $x \in K \setminus V$. By Corollary 2.6, I is an ideal of R'. Hence $R' \subseteq (I : I) = V$, and, since $x \notin V$, we have $x \notin R'$, whence $x^n \notin R$ for all $n \geq 1$. In particular, $x \in E(I)$, and we have $x^{-1}I \subseteq I$, i.e., $x^{-1} \in V$. Thus V is a valuation domain. Now let $P = \mathrm{Rad}(I)$. Then $IV_P \neq V_P$, so that $IV_P = I$ by Proposition 2.5. Since $(I : I)$ is the largest overring of R in which I is an ideal, we have that $V_P = V$, whence P is the maximal ideal of V.

(3) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (1). Statement (2) implies that I is strongly primary by Proposition 2.1. □

Corollary 2.12. *If R admits a nonzero proper principal strongly primary ideal, then R is a valuation domain.*

Proof. Let Ra be a nonzero principal strongly primary ideal of R. Then $R = (Ra : Ra)$ is a valuation domain by Theorem 2.11. □

Proposition 2.13. *Let I be a strongly primary ideal of R. Then*

(1) $I \subseteq xR$ for every $x \in R \setminus \mathrm{Rad}(I)$, and

(2) if I is finitely generated, then R is quasilocal with maximal ideal $\mathrm{Rad}(I)$.

Proof. Let $x \in R \setminus \mathrm{Rad}(I)$. Then $x \in E(I)$, and so (by Lemma 2.3) $x^{-1}I \subseteq I$. Hence $I \subseteq xI \subseteq xR$, proving (1). For (2), the relation $x^{-1}I \subseteq I$ shows that x^{-1} is integral over R; since $x \in R$, this implies that $x^{-1} \in R$. It follows that R is quasilocal with maximal ideal $\mathrm{Rad}(I)$. □

Proposition 2.14. *Let P be a strongly primary prime ideal of R, and let I be an ideal of R with $\mathrm{Rad}(I) = P$. Then PI is strongly primary. In particular, P^n is strongly primary for $n \geq 1$.*

Proof. Let $x \in E(PI)$. Since $P = \mathrm{Rad}(PI)$, we have $x \in E(P)$. Hence $x^{-1}P \subseteq P$, and we have $x^{-1}PI \subseteq PI$, as desired. □

3. ALMOST PSEUDO-VALUATION DOMAINS

Definition 3.1. *We say that a domain R is an almost pseudo-valuation domain (APVD) if every prime ideal of R is strongly primary.*

Recall from [6] that a prime ideal of a domain R is said to be *divided* if it is comparable to every ideal of R. If every prime of R is divided, then R is called a *divided domain*.

Proposition 3.2. *Let R be an APVD. Then R is a (quasilocal) divided domain. Moreover, every nonmaximal prime ideal of R is strongly prime.*

Proof. Proposition 2.13(1) shows that a strongly primary prime ideal is divided. The second statement follows from Theorem 2.8. □

Our next result shows that the requirement that each *primary* ideal of a domain be strongly primary is very restrictive (and hence explains why we defined APVD as we did). Recall that a prime ideal P of a domain R is said to be unbranched if P is the only P-primary ideal of R [9, p. 189]. A prime ideal P of a (pseudo-)valuation domain is unbranched $\Leftrightarrow$ P is the union of the chain of primes properly contained in P.

Proposition 3.3. *The following are equivalent for a domain R.*

(1) Each primary ideal of R is strongly primary.
(2) Either R is a valuation domain or R is a PVD with unbranched maximal ideal.

Proof. Suppose that each primary ideal of R is strongly primary. Then R is an APVD and is therefore quasilocal by Proposition 3.2. Assume that R is not a valuation domain, and let M be the maximal ideal of R. Then M is strongly primary, whence, by Theorem 2.11, $V = (M : M)$ is a valuation overring with M primary to the maximal ideal N of V. Suppose that $N \neq M$. Then by [9, Theorem 17.3], there is a prime ideal P of V with $\mathrm{ht}(N/P) = 1$. We must have $P \subseteq M$, so that P is also a prime ideal of R (possibly, $P = (0)$). Pick $a \in M \backslash P$. Then, since M is maximal in R, Ra is M-primary. Hence Ra is strongly primary. However, by Corollary 2.12, this implies that R is a valuation domain, a contradiction. Hence we must have $N = M$, and so R is a PVD. An argument similar to the one just given shows that M must be unbranched.

For the converse, recall that every primary ideal of a valuation domain is strongly primary by Proposition 2.1. Let (R, M) be a PVD with M unbranched, and let $V = (M : M)$ be the canonical valuation overring (also with maximal ideal M). Let I be a primary ideal of R. Since M is unbranched, $\mathrm{Rad}(I)$ is a prime ideal $P \neq M$. Since I is P-primary, we have $I = IR_P \cap R = IV_P \cap R \supseteq IV \cap R$. Hence $I = IV$. Thus $I = IV_P \cap V$, and I is primary to the prime ideal P in V. Hence I is strongly primary. □

Theorem 3.4. *The following statements are equivalent for a domain R.*

(1) R is an APVD.
(2) Some maximal ideal of R is strongly primary.
(3) If N is the set of nonunits of R, then $x^{-1}N \subseteq N$ *for each element* $x \in E(N)$.
(4) R is a quasilocal domain, and the maximal ideal M of R is such that $(M : M)$ *is a valuation domain with M primary to the maximal ideal of* $(M : M)$.
(5) R is a quasilocal domain, and there is a valuation overring in which M is a primary ideal.

Proof. The equivalence of (1) and (2) follows from Theorem 2.8, as does (2) ⇒ (3). To show that (3) ⇒ (2), it suffices to show that R is quasilocal with maximal ideal N. In fact, we claim that the prime ideals of R are linearly ordered. To see this, let P and Q be primes, and suppose that there is an element $b \in Q \backslash P$. Let $a \in P$. Then $b^n/a^n \notin R$ for each $n > 0$. Hence $b/a \in E(N)$, and we have $a/bN \subseteq N$ by assumption. In particular, $a^2/b \in N \subseteq R$, whence $a \in Q$. Thus $P \subseteq Q$, as desired. The equivalence of (1) and (4) follows from Theorems 2.8 and 2.11. Finally, it is clear that (4) ⇒ (5), and (5) ⇒ (2) by Proposition 2.1. □

Our next result shows that if R itself is strongly primary, then R is an APVD.

Proposition 3.5. *If R is strongly primary, then R is an APVD (and hence R admits a proper strongly primary ideal).*

Proof. Let M be a maximal ideal of R, and suppose that $xy \in M$ with $x, y \in K$ and $x \in E(M)$. We must show that $y \in M$. We may assume $xy \neq 0$. We have $x^n/xy \in E(R)$ for each $n \geq 1$. We first claim that $y \in R$. If $x \in E(R)$, then $y = x^{-1}xy \in R$, as claimed. On the other hand, if $x^k \in R$ for some k, then $(x^{k+1}/xy)y = x^k \in R$ with $x^{k+1}/xy \in E(R)$ again implies that $y \in R$. Now $(y^3/xy)(x^3/xy) = xy \in M$ with $x^3/xy \in E(R)$ implies that $y^3/xy \in R$, whence $y^3 = (y^3/xy)xy \in M$. Since $y \in R$, this yields $y \in M$, as desired. Thus M is strongly primary, and the result follows from Theorem 3.4. □

The converse of Proposition 3.5 is easily seen to be false. For example, if R is an integrally closed PVD which is not a valuation domain (e.g., $R = \mathbb{Q} + X\mathbb{Q}(Y)[[X]]$), then R is not strongly primary. (To see this, let V be the canonically associated valuation overring, and pick $u \in V \backslash R$. Then $u \in E(R)$, but $u^{-1} \notin R$.)

Corollary 3.6. *Let R be an APVD with maximal ideal M. If T is an overring of R with $MT = T$, then T is also an APVD.*

Proof. The hypotheses imply that T is strongly primary by Theorem 2.10. Hence T is an APVD by Proposition 3.5. □

Proposition 3.7. *If R is an APVD with maximal ideal M, then R' is a PVD with maximal ideal $N = \mathrm{Rad}(MR')$.*

Proof. By Corollary 2.6, $MR' = M$. Moreover, by Theorem 2.4, $\mathrm{Rad}(MR')$ is strongly prime. Hence $\mathrm{Rad}(MR')$ is the unique maximal ideal of R', and R' is a PVD. □

We close this section by considering domains each of whose overrings is an APVD. For PVD's there is a nice characterization: Every overring of a domain R is a PVD $\Leftrightarrow$ R' is a valuation domain [11, Proposition 2.7]. The situation with APVD's is not so clean.

Proposition 3.8. *If each overring of a domain R is an APVD, then R' is a valuation domain.*

Proof. We have that R' is a PVD by Proposition 3.7. The proof of [11, Proposition 2.7] shows that if R' is not a valuation domain, then there is a non-quasilocal overring of R'. However, such an overring cannot be an APVD by Proposition 3.2. □

The converse of Proposition 3.8 is false, as the following example shows.

Example 3.9. Let $V = k + N$ be a rank one discrete valuation domain with $N = Va$. Let $S = k + Va^2$, $T = k + Sa^2$, and $R = k + Va^4$. Then R is an APVD and $R' = V$, but T is not an APVD.

In Example 3.9, T is an integral overring of R. The following result shows that this is the only stumbling block.

Proposition 3.10. *Let R be an APVD with R' a valuation domain, and assume that every integral overring of R is an APVD. Then every overring of R is an APVD.*

Proof. Let T be a non-integral overring of R, and pick $x \in T\backslash R'$. If M is the maximal ideal of R, then Proposition 3.7 implies that $x^{-k} \in MR' = M$ for some $k \geq 1$. Hence x^{-k} is a unit of T, and we have $MT = T$. Therefore, T is an APVD by Corollary 3.6. □

4. CONDUCIVE DOMAINS

In [5, Theorem 4.5] Bastida and Gilmer prove that a domain R shares an ideal with a valuation overring $\Leftrightarrow$ each overring of R which is different from the quotient field K of R has a nonzero conductor to R. Domains with this property, called *conducive domains*, were explicitly defined and studied by Dobbs and Fedder [7] and further studied by Barucci, Dobbs, and Fontana [4, 8]. Conducive domains, powerful ideals, and strongly primary ideals are intimately connected, as we now show.

Theorem 4.1. *The following conditions are equivalent for a domain R.*

(1) R is a conducive domain.
(2) R admits a powerful ideal.
(3) R admits a strongly primary ideal.
(4) R shares a nonzero ideal with some conducive overring.

Proof. (1) $\Rightarrow$ (3): Since R is conducive, it follows from [4, Theorem 1] that R shares an ideal I with a valuation overring V such that I is primary in V. By Proposition 2.1, I is strongly primary.

(3) $\Rightarrow$ (2): If R admits a strongly primary ideal, then it admits a proper strongly primary ideal by Proposition 3.5. Hence R admits a powerful ideal by Corollary 2.6.

(2) $\Rightarrow$ (1): Let V be any valuation overring of R. Then by Proposition 1.17, R and V share an ideal. Hence R is conducive.

(1) $\Leftrightarrow$ (4): Assume (4). Let T be a conducive overring of R, and let V be a valuation overring of T. Then R and T share an ideal, and T and V share an ideal, from which it follows that R and V share an ideal. Thus R is conducive. (This argument is similar to that given in the proof of [7, Proposition 3.5], but the result was not stated explicitly there.) The converse is trivial. □

We close by using our methods to derive the characterizations of conducive Noetherian domains given in [4] and of conducive Prüfer domains given in [7].

Theorem 4.2 (cf. [4, Theorem 6]). *A Noetherian domain R is conducive $\Leftrightarrow$ each of the following conditions holds:*

(1) R is quasilocal of dimension one,
(2) R' is a rank one discrete valuation domain, and
(3) R' is a finitely generated R-module.

Proof. Suppose that R is conducive; then R admits a powerful ideal I. By Theorem 1.5, the primes contained in $P = \text{Rad}(I)$ are linearly ordered, and since R is Noetherian, this implies that $\mathrm{ht}P = 1$. Moreover, since P is contained in every nonzero prime of R, it follows that R is quasilocal with maximal ideal P. Now R' is a PVD by Theorem 1.15, whence R' is quasilocal and therefore a rank one discrete valuation domain [12, Exercise 14, p. 73]. Also, by Theorem 1.15, the conductor $J = (R : R')$ is nonzero, and since R is Noetherian, $J^{-1} = (R : J)$ is finitely generated. Thus it suffices to show that $J^{-1} = R'$. However, since R' is a rank one discrete valuation ring, we may write $J = xR'$. Hence $J^{-1} = x^{-1}(R : R') = x^{-1}xR' = R'$, as desired. For the converse, note that the fact that R' is a finitely generated R-module implies that R and R' share a common nonzero ideal. Hence the conclusion follows from Proposition 1.18. □

Let I be a nonzero powerful ideal of R. If $I = R$, then R is a valuation domain. If I is proper, then $\mathrm{Rad}(I)$ is contained in the Jacobson radical of R by Theorem 1.5, and $\mathrm{Rad}(I)$ is prime by Proposition 1.9. In either case, the Jacobson radical of R contains a nonzero prime ideal. For a Prüfer domain, the converse is true.

Theorem 4.3 (cf. [7, Corollary 3.4]). *Let R be a Prüfer domain (which is not a field). Then R is a conducive domain $\Leftrightarrow$ the Jacobson radical of R contains a nonzero prime ideal.*

Proof. Let Q be a nonzero prime ideal contained in the Jacobson radical of R. Let $x \in K\backslash R$; we shall show that $x^{-1}Q \subseteq R_M$ for each maximal ideal M of R. If $x \notin R_M$, this follows from the fact that $x^{-1} \in R_M$ (since R_M is a valuation domain). Suppose $x \in R_M$. If $x \notin QR_M$, then, since QR_M is strongly prime, the fact that $xx^{-1}Q \subseteq QR_M$ implies that $x^{-1}Q \subseteq QR_M \subseteq R_M$. Suppose that $x \in QR_M$, and choose $s \in R\backslash M$ with $sx \in Q$. Now, since $x \notin R$, we must have $x \notin R_N$ for some maximal ideal $N \neq M$. Then $x^{-1} \in R_N$, and hence $x^{-1}Q \subseteq QR_N$. It follows that $s = x^{-1}sx \in QR_N$, whence $s \in Q$. This contradiction completes the proof. □

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